

MAS212 Linear Algebra I 2007-2008

Coursework 1 — Solutions

Exercise* 1. Since $p(-1) = a - b + c$, $p(1) = a + b + c$, and $p(2) = 4a + 2b + c$, the 3×3 system for the coefficients a , b , and c is

$$\begin{array}{rrcr} a & - & b & + & c & = & 6 \\ a & + & b & + & c & = & 0 \\ 4a & + & 2b & + & c & = & 3 \end{array}$$

Exercise 2. In row echelon form: a, c, d, f, g; in reduced row echelon form: d, f.

Exercise 3.

(a) The corresponding linear system is

$$\begin{array}{rcl} x_1 & & = -1 \\ x_2 & & = 2 \\ x_3 & & = 3 \end{array}$$

Leading variables: x_1, x_2, x_3 ; no free variables (i.e. the solution is unique). We can directly read off the solution and find that the solution set is $\{(-1, 2, 3)\}$.

(b) The corresponding linear system is

$$\begin{array}{rcl} x_1 + 2x_2 & + & x_4 = 3 \\ x_3 + 2x_4 & = & 4 \end{array}$$

Leading variables: x_1, x_3 ; free variables: x_2, x_4 . Now set $x_2 = \alpha$ and $x_4 = \beta$. Using back substitution we find successively, $x_3 = 4 - 2x_4 = 4 - 2\beta$, and $x_1 = 3 - 2x_2 - x_4 = 3 - 2\alpha - \beta$. Hence the solution set is $\{(3 - 2\alpha - \beta, \alpha, 4 - 2\beta, \beta) : \alpha, \beta \in \mathbb{R}\}$.

(c) The corresponding linear system is

$$\begin{array}{rcl} x_1 & = & 0 \\ x_2 & = & 0 \\ 0 & = & 1 \end{array}$$

Leading variables: x_1, x_2 ; no free variables. Last line shows that system is inconsistent. Thus the solution set is empty.

(d) The corresponding linear system is

$$\begin{array}{rcl} x_2 & = & 2 \\ x_3 & = & -1. \end{array}$$

Leading variables: x_2, x_3 ; free variable: x_1 . Now set $x_1 = \alpha$, giving the solution set $\{(\alpha, 2, -1) : \alpha \in \mathbb{R}\}$.

Exercise* 4.

(a) The augmented matrix of the linear system is

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ -1 & 2 & 1 & -5 \\ 1 & 1 & 9 & 0 \end{array} \right).$$

(b) Using elementary row operations we find

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ -1 & 2 & 1 & -5 \\ 1 & 1 & 9 & 0 \end{array} \right) \xrightarrow[R_3 - R_1]{R_2 + R_1} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 7 & -3 \end{array} \right) \xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{array} \right),$$

where the last matrix is now in row echelon form.

(c) By (b), the linear system corresponding to the row echelon form of the original linear system is

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 3 \\ x_2 + 3x_3 &= -2 \\ x_3 &= 1 \end{aligned}$$

The leading variables are x_1 , x_2 , and x_3 ; there are no free variables. Using back substitution starting from the last equation we obtain successively $x_3 = 1$; then $x_2 = -2 - 3x_3 = -2 - 3 \cdot 1 = -5$; and finally $x_1 = 3 + x_2 - 2x_3 = 3 + (-5) - 2 \cdot 1 = -4$. Thus the solution set is $\{(-4, -5, 1)\}$, i.e. the solution is unique.

Exercise* 5.

(a) The augmented matrix of the system is

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 3 \\ 2 & 4 & 3 & 7 & 7 \\ -1 & -2 & 0 & 1 & -2 \end{array} \right).$$

(b) Using elementary row operations we find

$$\begin{aligned} &\left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 3 \\ 2 & 4 & 3 & 7 & 7 \\ -1 & -2 & 0 & 1 & -2 \end{array} \right) \xrightarrow[R_3 + R_1]{R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 & 1 \end{array} \right) \\ &\xrightarrow[R_3 - R_2]{R_1 - R_2} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right), \end{aligned}$$

where the last matrix is now in reduced row echelon form.

(c) By (b), the linear system corresponding to the reduced row echelon form of the original linear system is

$$\begin{aligned} x_1 + 2x_2 - x_4 &= 2 \\ x_3 + 3x_4 &= 1 \end{aligned}$$

The leading variables are x_1 and x_3 ; the free variables are x_2 , x_4 . No set $x_2 = \alpha$ and $x_4 = \beta$. Using back substitution starting from the last equation we obtain successively $x_3 = 1 - 3x_4 = 1 - 3\beta$, and then $x_1 = 2 - 2x_2 + x_4 = -2 - 2\alpha + \beta$. Thus the solution set is $\{(2 - 2\alpha + \beta, \alpha, 1 - 3\beta, \beta) : \alpha, \beta \in \mathbb{R}\}$.

Exercise 6. Applying elementary row operations to the augmented matrix of the system found in Exercise 1 gives:

$$\begin{aligned} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 1 & 1 & 1 & 0 \\ 4 & 2 & 1 & 3 \end{array} \right) & \xrightarrow[R_3 - 4R_1]{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 2 & 0 & -6 \\ 0 & 6 & -3 & -21 \end{array} \right) \xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 1 & 0 & -3 \\ 0 & 6 & -3 & -21 \end{array} \right) \\ & \xrightarrow{R_3 - 3R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & -3 & -3 \end{array} \right) \xrightarrow{-\frac{1}{3}R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array} \right), \end{aligned}$$

where the last matrix is now in row echelon form. The corresponding system reads

$$\begin{aligned} a - b + c &= 6 \\ b &= -3 \\ c &= 1 \end{aligned}$$

so $c = 1$, $b = -3$, and $a = 6 + b - c = 2$. Thus the polynomial is $p(x) = 2x^2 - 3x + 1$.

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